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NOTE

On the Delone Triangulation Numbers

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"Almost exact" estimates for the Delone triangulation numbers are given. In particular

$$\lim_{n \to \infty} \frac{\mathrm{del}_{n-1}}{n} = \frac{1}{2\pi e} = 0.0585498....$$

By B_2^n we denote the Euclidean ball in $\mathbb{R}^n$. Recall that the symmetric difference metric for convex bodies K and C is the volume of the difference of K and C

$$\mathrm{d}_S(K, C) = \mathrm{vol}_n(K \triangle C).$$

McClure and Vitale, [McVi], in dimension 2 and Gruber, [Gr], in arbitrary dimension obtained an asymptotic formula for convex bodies K

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in $\mathbb{R}^n$ with a C^2-boundary with everywhere positive curvature. Namely, for such bodies

$$\lim_{N\to\infty} N^{2/(n-1)} \inf\{\mathrm{d}_S(K, P_N) \mid P_N \subset K, P_N \text{ is a polytope with } N \text{ vertices}\}$$

is equal to

$$\tfrac{1}{2}\,\mathrm{del}_{n-1}\left(\int_{\partial K} \kappa(x)^{1/(n+1)}\, d\mu(x)\right)^{(n+1)/(n-1)},$$

where μ is the surface measure, κ the Gauss-curvature, and del_{n-1} is a constant connected with the Delone triangulations. Thus

$$\mathrm{del}_{n-1} = \lim_{N\to\infty} \frac{\begin{pmatrix} 2\inf\{\mathrm{d}_S(K, P_N) \mid P_N \subset K, \\ \qquad P_N \text{ is a polytope with } N \text{ vertices}\} \end{pmatrix}}{\left(\int_{\partial K} \kappa(x)^{1/(n+1)}\, d\mu(x)\right)^{(n+1)/(n-1)} N^{-2/(n-1)}}.$$

In particular, for $K = B_2^n$, we get

$$\mathrm{del}_{n-1} = \lim_{N\to\infty} \frac{\begin{pmatrix} 2\inf\{\mathrm{d}_S(B_2^n, P_n) \mid P_N \subset B_2^n, \\ \qquad P_N \text{ is a polytope with } N \text{ vertices}\} \end{pmatrix}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)}\, N^{-2/(n-1)}}.$$

It was shown in [GRS] that there are constants c_1 and c_2 such that

$$c_1 n \leqslant \mathrm{del}_n \leqslant c_2 n.$$

This result was refined in [MS]:

$$\frac{n-1}{n+1}\,\mathrm{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)} \leqslant \mathrm{del}_{n-1} \leqslant 2^{0.802}\,\mathrm{vol}_{n-1}(\partial B_2^n)^{-2/(n-1)}.$$

Let K be a convex body. We consider random polytopes with vertices (randomly) chosen from the boundary of the body K. The expected volume of such a random polytope is defined by

$$\mathbb{E}(\partial K, N) = \int_{\partial K} \cdots \int_{\partial K} \mathrm{vol}_n([x_1, ..., x_N])\, d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N),$$

where $\mathbb{P}$ denotes the normalized surface measure on the boundary of K and $[x_1, ..., x_N]$ denotes the convex hull of the points $x_1, ..., x_N$.

PROPOSITION 1 (J. Müller, [Mü], Theorem 2). *In the above notation, for* $K = B_2^n$ *one has*

$$\lim_{N\to\infty} \frac{\operatorname{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N)}{N^{-2/(n-1)}}$$

$$= \frac{\operatorname{vol}_{n-2}(\partial B_2^{n-1})}{2(n+1)!} \left(\frac{(n-1)\operatorname{vol}_{n-1}(\partial B_2^n)}{\operatorname{vol}_{n-2}(\partial B_2^{n-1})} \right)^{(n+1)/(n-2)} \Gamma\left(n+1+\frac{2}{n-1}\right).$$

The following theorem provides "almost exact" estimates for the Delone triangulation numbers

THEOREM 2. *For every* $n \in \mathbb{N}$ *with* $n \geqslant 2$

$$\frac{n-1}{n+1}\operatorname{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)}$$

$$\leqslant \operatorname{del}_{n-1} \leqslant \frac{n-1}{n+1}\operatorname{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!}.$$

Consequently there is a numerical constant $c > 0$ *such that*

$$\frac{n-1}{n+1}\operatorname{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)}$$

$$\leqslant \operatorname{del}_{n-1} \leqslant \left(1+\frac{c\ln n}{n}\right)\frac{n-1}{n+1}\operatorname{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)}.$$

In particular,

$$\lim_{n\to\infty} \frac{\operatorname{del}_{n-1}}{n} = \frac{1}{2\pi e}.$$

Proof. The left-hand inequality was shown in [MS], Theorem 3. To prove the right hand side inequality note that

$$\operatorname{del}_{n-1} = \lim_{N\to\infty} \frac{\begin{pmatrix} 2\inf\{\mathrm{d}_S(B_2^n, P_N) \mid P_N \subset B_2^n, \\ \qquad P_N \text{ is a polytope with } N \text{ vertices}\} \end{pmatrix}}{(\operatorname{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)} N^{-2/(n-1)}}$$

$$\leqslant \lim_{N\to\infty} \frac{2(\operatorname{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N))}{(\operatorname{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)} N^{-2/(n-1)}}.$$

By Proposition 1 we have

$$\mathrm{del}_{n-1} \leqslant \frac{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}{(n+1)!} \left(\frac{n-1}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})} \right)^{(n+1)/(n-1)} \Gamma\left(n+1+\frac{2}{n-1}\right)$$

$$= \frac{n-1}{n+1} (\mathrm{vol}_{n-2}(B_2^{n-1}))^{-2/(n-1)} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!}.$$

By an elementary calculation

$$\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!} \leqslant 1 + \frac{c \ln(n+2)}{n}.$$

It remains to show that

$$\lim_{n\to\infty} \frac{\mathrm{del}_{n-1}}{n} = \frac{1}{2\pi e}.$$

This follows from

$$\lim_{n\to\infty} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)}}{n} = \frac{1}{2\pi e}. \quad \blacksquare$$

REFERENCES

[GRS] Y. Gordon, S. Reisner, and C. Schütt, Umbrellas and polytopal approximation of the Euclidean ball, *J. Approx. Theory* **90** (1997), 9–22. Erratum, *J. Approx. Theory* **95** (1998), 331.

[Gr] P. M. Gruber, Asymptotic estimates for best and stepwise approximation of convex bodies II, *Forum Math.* **5** (1993), 521–538.

[MS] P. Mankiewicz and C. Schütt, A simple proof of an estimate for the approximation of the Euclidean ball and the Delone triangulation numbers, *J. Approx. Theory* **107** (2000), 268–280.

[McVi] McClure and R. Vitale, Polygonal approximation of plane convex bodies, *J. Math. Anal. Appl.* **51** (1975), 326–358.

[Mü] J. S. Müller, Approximation of the ball by random polytopes, *J. Approx. Theory* **63** (1990), 198–209.

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